b-Metric Generalization of Some Fixed Point Theorems

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Abstract

We Show that common fixed point theorems in terms of b-metric spaces with new contraction mapping have unique fixed point. Further, we include b-metric generalizations of some fixed point theorems of Fisher, Pachpatte, and Sahu and Sharma.

Keywords : b – Metric Space, Contraction, Fixed Point.

1 Introduction

Fixed point theory is a fascinating topic for research inboth analysis and topology. In this direction the Banach contraction mapping theorem of 1922 popularly known as Banach contraction mapping principle is a rewarding result in fixed point theory. It has widespread applications in both pure and applied mathematics. The well-known Banach [1] contraction mapping principle states that if "X is a complete metric space and $T : X \rightarrow X$is a contraction mapping of X into itself then T has unique fixed point in X." This celebrated principle has been generalized by several authors. In 1989, Bakhtin [2] introduced the concept of b-metric space which is generalization of renowned Banach contraction mapping principle. Czerwik [3, 4] extended the concept of b-metric space in 1993. Bakhtin's concept of b-metric spaces has been extensively generalized and improved by several mathematicians for fixed points in several different ways, namely, Boriceanu [5], Bota et al. [6], Chen et al. [7], Hussain and Shah [8], Kutbi et al. [9], and Shukla [10] to name a few. In this paper, our main concern is to study common fixed point theorems in complete b-metric spaces for three self-mappings. The obtained results are generalizations of b- metric variant of fixed point theorems of Fisher, Pachpatte, and Sahu and Sharma.

The following fixed point theorems were proved in [11– 13].

Theorem 1 [11] Let T be a mapping of the complete metric space X into itself satisfying the inequality

$$[d(Ta,Tb)]^2 \le a_1[d(a,Ta)d(b,Tb)] + a_2[d(a,Tb)d(b,Ta)],$$

$\forall\, a,b \in X, 0 \le a_1 < 1, 0 \le a_2$, then T has a fixed point in X.

Theorem 2 [12] Let T is a mapping of the complete metric space X into itself satisfying the inquality

$$[d(Ta,Tb)]^2 \le a_1[d(a,Ta)d(b,Tb) + d(a,Tb)d(b,Ta)] + a_2[d(a,Ta)d(b,Ta) + d(a,Tb)d(b,Tb)],$$

$\forall a,b \in X,\ where\ a_1, a_2 \ge 0\, and\ a_1 + a_2 < 1,\ then\, T\ has\ a\ unique\ \ fixed\ \ point\ \ in\ X.$

Theorem 3 [13] Let T is a mapping of the complete metric space X into itself satisfying the inequality.

$$\begin{aligned}[d(Ta,Tb)]^2 \le\ & a_1[d(a,Ta)d(b,Tb) + d(a,Tb)d(b,Ta)] \\ & + a_2[d(a,Ta)d(b,Tb) + d(a,Tb)d(b,Ta)] \\ & + a_3[\{d(b,Ta)\}^2 + \{d(b,Tb)\}^2]\end{aligned}$$

$\forall a,b \in X\ , where\ a_1, a_2, a_3 \ge 0\, and\ a_1 + 2a_2 + a_3 < 1\ then\, T\ has\ a\ unique\ \ fixed\ \ point\ \ in\ X.$

2. Preliminaries

In this section we recall some basic definitions and necessary results from existing literature that will be used in the sequel.

Definition 4 [3]. Let X be a nonempty set and $s \ge 1$ be a given real number. A function $d : X \times X \to R+$ is said to be a b-metric on X if the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, y) \le s(d(x, z) + d(z, y))$ for all $x, y, z \in X$.

The pair (X, d) is called b-metric space.

It is clear from the definition of b-metric that every metric space is b-metric for $s = 1$, but the converse need not be true.

The following example illustrates the above remarks.

Example 5 [5]. Let $X = \{0, 1, 2\}$. Define $d : X \times X \to R+$ by

$d(0, 0) = d(1, 1) = d(2, 2) = 0$, $d(1, 2) = d(2, 1) = d(1, 0) = d(0, 1) = 1$, and

$d(0, 2) = d(2, 0) = m \geq 2$ for $s = m/2$.

The function defined above is a b-metric space but is not a metric space for $m > 2$.

Proposition 6 [16] Let be a non empty set and the mapping $T,g,h:X\to X$ have a unique point of coincidence in X. If {T, h} and {g, h} are weakly compatible self-maps of X ,then T, g, h have a unique fixed point.

Definition 7. A sequence $\{xn\}$ in a b-metric space (X, d) is called Cauchy sequence if and only if $\lim_{m,n\to\infty} d(xn, xm) = 0$.

Definition 8. A sequence $\{xn\}$ in a b-metric space (X, d) is said to converge to a point $x \in X$ if and only if $\lim_{n\to\infty} d(xn, x) = 0$.We denote this by $\lim_{n\to\infty} xn = x$.

Definition 9. A b-metric space (X, d) is said to be complete if and only if every Cauchy sequence in X converges to a point of X.

Definition 10 [15]. Let f and g be self-mappings of a set X. If $z = fx = gx$ for some x in X, then x is called a coincidence point of f and g and z is called a point of coincidence of f and g.

Definition 11 [16]. The mappings $f,g : X \to X$ are weakly compatible, if, for every $x \in X$, the following holds: $f(gx) = g(fx)$ whenever $fx = gx$.

Definition 12. A point $x \in X$ is said to be a fixed point of a self-map $T : X\to X$ if $T(x) = x$.

3. Main Results

In this section we obtain coincidence points and common fixed point theorems for three maps in complete b-metric spaces. In order to start our main results we begin with a simple but useful Lemma.

Lemma 13. Let (X,d) be a complete b-metric space with the constant coefficient $s \geq 1$ *and let* $T, g, h : X \to X$ be self – mapping from X into itself satisfying the following conditions:

i. $T(X) \cup g(X) \subseteq h(X)$

ii.

$$
\begin{aligned}
[d(Ta,gb)]^2 \leq\ & a_1[d(ha,Ta)d(hb,gb)+d(hb,hb)+d(ha,gb)d(hb,Ta)] \\
& +a_2[d(hb,ga)d(hb,Ta)+d(ha,gb)d(hb,gb)] \\
& +a_3[\{d(hb,Ta)\}^2\{d(hb,gb)\}^2 + a_4[\frac{d(hb,gb)}{1+d(hb,Ta)}]^2
\end{aligned}
\quad \text{---------(3.1)}
$$

$\forall\, a,b \in X \;\; ; a_1, a_2, a_3, a_4 \geq 0, \ni : sa_1 + (s^2 + s)a_2 + a_3 + a_4 < 1.$

Then every sequence $\{b_n\}$with initial point a_0 is a cauchy's sequence in X.

Proof: Let $a_0 \in X$ and choose a point $a_1 \in X$ such that $ha_1=Ta_0$ and for a_1 there exist $a_2 \in X$

Such that $ha_2=ga_1$, continuing this process we construct sequence $\{a_n\}$ and $\{b_n\}$ in X given by

$$b_{2n} = ha_{2n+1} = Ta_{2n}$$
$$b_{2n+1} = ha_{2n+2} = ga_{2n+1} \;\; \forall n \geq 0$$

Suppose that there exists $h \in [0,1/s)$ such that

$$d(b_n, b_{n+1}) \leq hd(b_{n-1}, b_n) \quad \forall n \geq 1.$$

We show that $\{b_n\}$ is a cauchy sequence in X . Using (3.1), we have

$$\begin{aligned}
[d(ba_{2n}, b_{2n+1})]^2 &= [d(ha_{2n+1}, ha_{2n+2}]^2 \\
&= [d(Ta_{2n}, ga_{2n+1)}]^2 \\
&\leq a_1[d(ha_{2n}, Ta_{2n})d(ha_{2n+1}, ga_{2n+1}) + d(ha_{2n}, ga_{2n+1})d(ha_{2n+1}, Ta_{2n})] \\
&\quad + a_2[d(ha_{2n+1}, ga_{2n})d(ha_{2n+1}, Ta_{2n}) + d(ha_{2n}, ga_{2n+1})d(ha_{2n+1}, ga_{2n+1})] \\
&\quad + a_3[\{d(ha_{2n+1}, Ta_{2n})\}^2 + \{d(ha_{2n+1}, ga_{2n+1})\}]^2 \\
&\quad + a_4[\frac{d(ha_{2n+1}, ga_{2n+1})}{1 + d(ha_{2n+1}, Ta_{2n})}]^2 \\
&\leq a_1[d(ha_{2n}, ha_{2n+1})d(ha_{2n+1}, ha_{2n+2}) \\
&\quad + a_2[d(ha_{2n+1}, ha_{2n+1})d(ha_{2n+1}, ha_{2n+1}) \\
&\quad + d(ha_{2n}, ha_{2n+2})d(ha_{2n+1}, ha_{2n+2})] \\
&\quad + a_3[\{d(ha_{2n+1}, ha_{2n+1})\}^2 + \{d(ha_{2n+1}, ha_{2n+2})\}^2] \\
&\quad + a_4[\frac{d(ha_{2n+1}, ha_{2n+2})}{1 + d(ha_{2n+1}, ha_{2n+1})}]^2
\end{aligned}$$

$$\begin{aligned}
&\leq a_1[d(b_{2n-1}, b_{2n})d(b_{2n}, b_{2n+1}) + d(b_{2n-1}, b_{2n+1})d(b_{2n}, b_{2n})] \\
&\qquad + a_2[d(b_{2n}, b_{2n})d(b_{2n}, b_{2n}) + d(b_{2n+1}, b_{2n-1})d(b_{2n}, b_{2n+1})] \\
&\qquad + a_3[\{d(b_{2n}, b_{2n})\}^2 + \{d(b_{2n}, b_{2n+1})\}^2] + a_4[\frac{d(b_{2n}, b_{2n+1})}{1 + d(b_{2n}, b_{2n})}]^2
\end{aligned}$$

$$
\begin{aligned}
&\le a_1[d(b_{2n-1},b_{2n})d(b_{2n},b_{2n+1})+a_2[d(b_{2n-1},b_{2n+1})d(b_{2n},b_{2n+1})] \\
&\quad +a_3[\{d(b_{2n},b_{2n+1})+a_4 d(b_{2n},b_{2n+1})]^2 \\
&\le d(b_{2n},b_{2n+1})[a_1 d(b_{2n-1},b_{2n})+a_2 d(b_{2n-1},b_{2n+1}) \\
&\quad +a_3 d(b_{2n},b_{2n+1})+a_4 d(b_{2n},b_{2n+1})] \\
&\le a_1 d(b_{2n-1},b_{2n})+sa_2[d(b_{2n-1},b_{2n})+d(b_{2n},b_{2n+1})] \\
&\quad +a_3 d(b_{2n},b_{2n+1})+a_4 d(b_{2n},b_{2n+1}) \\
&\le (a_1+sa_2)d(b_{2n-1},b_{2n})+(sa_2+a_3+a_4)d(b_{2n},b_{2n+1})
\end{aligned}
$$

$$[1-(sa_2+a_3+a_4)]d(b_{2n},b_{2n+1})\le (a_1+sa_2)d(b_{2n-1},b_{2n})$$

$$d(b_{2n},b_{2n+1})\le \frac{a_1+sa_2}{[1-(sa_2+a_3+a_4)]}d(b_{2n-1},b_{2n})$$

$$d(b_{2n},b_{2n+1})\le \lambda d(b_{2n-1},b_{2n}),$$

$$where\ \lambda=\frac{a_1+sa_2}{[1-(sa_2+a_3+a_4)}<\frac{1}{s}<1,$$

$$d(b_{2n},b_{2n+1})\le \lambda d(b_{2n-1},b_{2n})$$

$$d(b_{2n+1},b_{2n+2})\le \lambda d(b_{2n},b_{2n+1}).$$

$\therefore \forall\ n\in N, we\ write$

$$d(b_{2n+1},b_{2n+2})\le \lambda d(b_n,b_{n+1})\le\cdots\le \lambda^{n+1}d(b_0,b_1)$$

Now , for any m,n$\in$N,m>n, we have

$$
\begin{aligned}
d(b_n,b_m) &\le sd(b_n,b_{n+1})+sd(b_{n+1},b_m) \\
&\le sd(b_n,b_{n+1})+s^2 d(b_{n+1},b_{n+2})+s^2 d(b_{n+2},b_m) \\
&\le sd(b_n,b_{n+1})+s^2 d(b_{n+1},b_{n+2})+s^3 d(b_{n+2},b_{n+3})+\cdots+s^{m-n-1}d(b_{m-2},b_{m-1}) \\
&\qquad +s^{m-n}d(b_{m-1},b_m) \\
&\le [s\lambda^n+s^2\lambda^{n+1}+\cdots+s^{m-n}\lambda^{m-1}]d(b_0,b_1) \\
&\le \frac{s\lambda^n}{(1-s\lambda)}d(b_0,b_1).
\end{aligned}
$$

Therefore ,we have

$$d(b_n,b_m)\le \frac{s\lambda^n}{(1-s\lambda)}d(b_0,b_1)\to 0\ \ as\ n\to\infty$$

Thus,

$$d(b_n, b_m) \to 0 \ as \ n \to \infty$$

Hence $\{b_n\}$ is a Cauchy sequence in b-metric space X.

The next theorem is *b*-metric variant of Theorem 1.3 in [13].

Theorem 14. *Let* (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and suppose that the self-maps $T, g, h: X \to X$ satisfy the conditions

$$\begin{aligned}[d(Ta, gb)]^2 \leq\ & a_1[d(ha, Ta)d(hb, gb) + d(ha, gb)d(hb, Ta)] \\ & + a_2[d(hb, ga)d(hb, Ta) + d(ha, gb)d(hb, gb)] \\ & + a_3[\{d(hb, Ta)\}^2 + \{d(hb, gb)\}^2] + a_4[\frac{d(hb, gb)}{1 + d(hb, Ta)}]^2 \end{aligned} \quad \text{---------------(3.2)}$$

For all $a, b \in X$; $a_1, a_2, a_3, a_4 \geq 0$, are nonnegative reals with

$$sa_1 + (s^2 + s)a_2 + a_3 + a_4 < 1.$$

If $T(X) \cup g(X) \subseteq h(X)$ is a complete subspace of X , then the maps T,g and h have a coincidence point v in X. Moreover, if {T,h} and {g,h} are weakly compatible Pairs. Then T,g and h have a unique common fixed point in X.

Proofs: Let a_0 be an arbitrary point in X and define the sequence $\{b_n\}$ in X such that

$$\begin{aligned} b_{2n} &= ha_{2n+1} = Ta_{2n} \\ b_{2n+1} &= ha_{2n+2} = ga_{2n+1} \quad \forall n \geq 0 \end{aligned}$$

Now , we show that $\{b_n\}$ is a cauchy sequence.so by (4), we have

$$[d(ba_{2n}, b_{2n+1})]^2 = [d(ha_{2n+1}, ha_{2n+2}]^2$$

$$\begin{aligned} &= [d(Ta_{2n}.ga_{2n+1})]^2 \\ &\leq a_1[d(ha_{2n}, Ta_{2n})d(ha_{2n+1}, ga_{2n+1}) + d(ha_{2n}, ga_{2n+1})d(ha_{2n+1}, Ta_{2n})] \\ &+ a_2[d(ha_{2n+1}, ga_{2n})d(ha_{2n+1}, Ta_{2n}) + d(ha_{2n}, ga_{2n+1})d(ha_{2n+1}, ga_{2n+1})] \\ &+ a_3[\{d(ha_{2n+1}, Ta_{2n})\}^2 + \{d(ha_{2n+1}, ga_{2n+1})\}^2] + a_4\left[\frac{d(ha_{2n+1}, ga_{2n+1})}{1 + d(ha_{2n+1}, Ta_{2n})}\right]^2 \end{aligned}$$

$$\leq a_1[d(ha_{2n}, ha_{2n+1})d(ha_{2n+1}, ha_{2n+2}) + d(ha_{2n}, ha_{2n+2})d(ha_{2n+1}, ha_{2n+1})]$$
$$+ a_2[d(ha_{2n+1}, ha_{2n+1})d(ha_{2n+1}, ha_{2n+1}) + d(ha_{2n}, ha_{2n+2})d(ha_{2n+1}, ha_{2n=2})]$$
$$+ a_3[\{d(ha_{2n+1}, ha_{2n+1})\}^2 + \{d(ha_{2n+1}, ha_{2n+2})\}^2] + a_4[\frac{d(ha_{2n+1}, ha_{2n+2})}{1 + d(ha_{2n+1}, ha_{2n+1})}]^2$$

$$\leq a_1\left[d(b_{2n-1}, b_{2n})d(b_{2n}, b_{2n+1}) + d(b_{2n-1}, b_{2n+1})d(b_{2n}, b_{2n})\right] + a_2[d(b_{2n}, b_{2n})d(b_{2n}, b_{2n}) +$$
$$d(b_{2n-1}, b_{2n+1})d(b_{2n}, b_{2n+1})] + a_3\left[\{d(b_{2n}, b_{2n})\}^2 + \{d(b_{2n}, b_{2n+1})\}^2\right] + a_4[\frac{d(b_{2n}, b_{2n+1})}{1 + d(b_{2n}, b_{2n})}]^2$$
$$\leq a_1[d(b_{2n-1}, b_{2n})d(b_{2n}, b_{2n+1})] + a_2\left[d(b_{2n-1}, b_{2n+1})d(b_{2n}, b_{2n+1})\right]$$
$$+ a_3\left[\{d(b_{2n}, b_{2n+1})\}^2\right] + a_4[d(b_{2n}, b_{2n+1})]^2$$

$$\leq d(b_{2n}, b_{2n+1})[a_1 d(b_{2n-1}, b_{2n}) + a_2 d(b_{2n-1}, b_{2n+1}) + a_3 d(b_{2n}, b_{2n+1}) + a_4 d(b_{2n}, b_{2n+1})]$$
$$\leq a_1[d(b_{2n-1}, b_{2n})] + sa_2[d(b_{2n-1}, b_{2n}) + d(b_{2n}, b_{2n+1})] + a_3 d(b_{2n}, b_{2n+1}) + a_4 d(b_{2n}, b_{2n+1})$$
$$\leq (a_1 + sa_2)d(b_{2n-1}, b_{2n}) + (sa_2 + a_3 + a_4)d(b_{2n}, b_{2n+1})$$

$$[1 - (sa_2 + a_3 + a_4)]d(b_{2n}, b_{2n+1}) \leq (a_1 + sa_2)d(b_{2n-1}, b_{2n})$$
$$d(b_{2n}, b_{2n+1}) \leq \frac{a_1 + sa_2}{[1 - (sa_2 + a_3 + a_4)} d(b_{2n-1}, b_{2n})$$
$$d(b_{2n}, b_{2n+1}) \leq \lambda d(b_{2n-1}, b_{2n}),$$

$$where \quad \lambda = \frac{a_1 + sa_2}{[1 - (sa_2 + a_3 + a_4)]} < \frac{1}{s} < 1,$$
$$d(b_{2n}, b_{2n+1}) \leq \lambda d(b_{2n-1}, b_{2n})$$

Similarly ,we can show that ,

$$d(b_{2n+1}, b_{2n+2}) \leq \lambda d(b_{2n}, b_{2n+1})$$

Therefore, for all $m, n \in N \quad with \; m > n,$ we can get

$$d(b_{n+1}, b_{n+2}) \leq \lambda d(b_n, b_{n+1}) \leq ..... \leq \lambda^{n+1} d(b_0, b_1)$$

Fix $m > n; m, n \in \mathrm{N},$ we get

$$d(b_n, b_m) \le sd(b_n, b_{n+1}) + sd(b_{n+1}, b_m)$$
$$\le sd(b_n, b_{n+1}) + s^2 d(b_{n+1}, b_{n+2}) + s^2 d(b_{n+2}, b_m)$$
$$\le sd(b_n, b_{n+1}) + s^2 d(b_{n+1}, b_{n+2}) + s^3 d(b_{n+2}, b_{n+3})$$
$$+ \ldots.. + \ s^{m-n-1} d(b_{m-2}, b_{m-1}) + s^{m-n} d(b_{m-1}, b_m)$$
$$\le [s\lambda^n + s^2 \lambda^{n+1} s^3 \lambda^{n+2} + \ldots. + s^{m-n} \lambda^{m-1}] d(b_0, b_1)$$
$$\le \frac{s\lambda^n}{(1 - s\lambda)} d(b_0, b_1).$$

Thus, as $n \to \infty, d(b_n, b_m) \to 0$. It follows from lemma 4.2.1 that $\{b_n\}$ is a Cauchy sequence and ,by the completeness of X, $\{b_n\}$ converges to same $b \in X$. Therefore,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} Ta_{2n} = \lim_{n\to\infty} ga_{2n+1} = \lim_{n\to\infty} ha_{2n+1} = \lim_{n\to\infty} ha_{2n+2} = b$$

Thus , $T(X) \bigcup g(X) \subseteq h(X)$ implies either $T(X) \subseteq h(X)$ *or* $g(X) \subseteq h(X)$.

Case1. Let $T(X) \subseteq h(X)$. since h(x) is a complete subspace of *and* $T(X) \subseteq h(X)$ *implies* $h(x)$

is closed,hence,there exist $u, b \in X$

Such that hu=b. if $gv \ne b,$ then, by using 14 we get

$$[d(Ta_{2n}, gv)]^{2,} \le a_1, [d(ha_{2n}, Ta_{2n}) d(hu, gv) + d(ha_{2n}, gv) d(hu, Ta_{2n})]$$
$$+ a_2 [d(hu, ga_{2n}) d(hu, Ta_{2n}) + d(ha_{2n}, gv) d(hv, gv)]$$
$$+ a_3 [\{d(hv, Ta_{2n})\}^2 + \{d(hv, gv)\}^2] + a_4 [\frac{d(hv, gv)}{1 + d(hv, Ta_{2n})}]$$

Taking limit as $n \to \infty$ yields

$$[d(b, gv)]^2 \le a_1 [d(b, b) d(b, gv) + d(b, gv) d(b, b)$$
$$+ a_2 [d(b, b) d(b, b) + d(b, gv) d(b, gv)$$
$$+ a_3 [\{d(b, b)\}^2 + \{d(b, gv)\}^2] + a_4 [\left\{\frac{d(b, gv)}{1 + d(b, b)}\right\}^2]$$
$$\le (a_2 + a_3 + a_4)[d(b, gv)]^2$$
$$\Rightarrow [1 - (a_2 + a_3 + a_4)][d(b, gv)^2] \le 0$$

And the above inequality is possible only if $[d(b, gv)^2 = 0$ *which implies that* $b = gv.$ it follows that

$$hv = b = av$$

Since g and h are weakly compatible, we have $hgv = ghv$ and so

$$gb = hp$$

If $b \neq gb$,then by 3.2 we have

$$\left[d\left(Ta_{2n}, gb\right)\right]^2 \leq a_1[d\left(ha_{2n}, Ta_{2n}\right)d\left(hb, gb\right) + d\left(ha_{2n}, gb\right)d\left(hb, Ta_{2n}\right)]$$
$$+ a_2[d,\left(hb, ga_{2n}\right)d\left(hb, Ta_{2n}\right) + d\left(ha_{2n}, gb\right)d\left(hb, gb\right)]$$
$$+ a_3[\left\{d\left(hb, Ta_{2n}\right)\right\}^2 + \left\{d\left(hb, gb\right)\right\}^2] + a_4[\frac{d\left(hb, gb\right)}{1 + d\left(hb, Ta_{2n}\right)}]^2$$

$As\ n \to \infty, we\ \ have$

$$[d\left(b, gb\right)^2] \leq a_1[d\left(b, b\right)d\left(hb, gb\right) + d\left(b, gb\right)d\left(hb, b\right)$$

$$+ a_2\left[d(hb, b)d(hb, b) + d(b, gb)d(hb, gb)\right] +$$

$$a_3\left[\left\{d(hb, b)\right\}^2 + \left\{d(hb, gb)\right\}^2\right] + a_4\left[\frac{d(hb, gb)}{1 + d(hb, b)}\right]^2$$

$$\leq a_1\left[d(b, b)d(gb, gb) + d(b, gb)d(gb, b)\right] + a_2\left[d(gh, b)d(gb, b) + d(b, gb)d(gb, gb)\right]$$

$$+ a_3\left[\left\{d(gb, b)\right\}^2 + \left\{d(gb, gb)\right\}^2\right] + a_4\left[\frac{d(gb, gb)}{1 + d(hb, b)}\right]^2$$

$$\leq (a_1 + a_2 + a_3)\left[\left\{d(b, gb)\right\}^2\right]$$

$$\Rightarrow \left[1 - (a_1 + a_2 + a_3)\right]\left[d(b, gb)^2\right] \leq 0,$$

And the above inequality is possible only if

$$\left[d(b, gb)\right]^2 = 0 \Rightarrow b = gb$$

By using 3.3, we have

hb = gb=b. -- (3.4)

Case 2:

If g(X)$\subseteq$h(X), again there exists points v, b$\in$X such that b = hv if b$\neq$Tv , and then by using 4.2 we get

$$\left[d(Tv, gb)\right]^2 \leq a_1\left[d(hv, Tv)d(hb, gb) + d(hv, gb)d(hb, Tv)\right]$$

$$+a_2\left[d(hb,gb)d(hb,Tv)+d(hb,gb)d(hb,gb)\right]$$

$$+a_3\left[\{d(hb,Tv)\}^2+\{d(hb,gb)\}^2\right]+a_4\left[\frac{d(hb,gb)}{1+d(hb,Tv)}\right]^2$$

It follows that,

$$\left[d(Tv,b)\right]^2 \le a_1\left[d(b,Tv)d(b,b)+d(b,b)d(b,Tv)\right]+$$
$$a_2\left[d(b,b)d(b,Tv)+d(b,b)d(b,b)\right]+$$
$$a_3\left[\{d(b,Tv)\}^2+\{d(b,b)\}^2\right]+a_4\left[\frac{d(b,b)}{1+d(b,Tv)}\right]^2$$
$$\le a_3\left[d(b,Tv)\right]^2$$
$$\Rightarrow \left[1-a_3\right]\left[d(b,Tv)\right]^2 \le 0$$

Which is possible only if

$$d(b,Tv)=0 \Rightarrow b=Tv.$$
$$\Rightarrow Tv=hv=b$$

Since T and h weakly compatible and hence Thv=hTv and so

Tb=hb

and by 3. 4 we have

b=Tb=gb=hb.

Thus, b is the common fixed point of self –mappings T, g and h. This completes the proof of the theorem.

Uniqueness. In order to prove uniqueness, let $b_1 \neq b_2$ be two distinct common fixed points of the self –maps T, g and h then we have by 3. 2

$$\left[d(b_1 b_2)\right]^2=\left[d(Tb_1,gb_2)\right]^2$$

$$\le a_1\left[d(hb_1,Tb_1)d(hb_2,gb_2)+d(hb_1,gb_2)d(hb_2,Tb_1)\right]+$$
$$a_2\left[d(hb_2,gb_1)d(hb_2,Tb_1)+d(hb_1,gb_2)d(hb_2,gb_2)\right]+$$
$$a_3\left[\{d(hb_2,Tb_1)\}^2+\{d(hb_2,gb_2)\}^2\right]+a_4\left[\frac{d(hb_2,gh_2)}{1+d(hb_2,Tb_1)}\right]^2$$

$$\leq \; a_1[d(b_1,b_1)d(b_2,b_2)+d(b_1,b_2)d(b_2,b1)]+ \quad a_2[d(b_2,b_1)d(b_2,b_1)+d(b_1,b_2)d(b_2,b_2)]$$

$$+a_3[\{d(b_2,b_1)\}^2+\{d(b_2,b_2)\}^2]+a_4\left[\frac{d(b_2,b_2)}{1+d(b_2,b_1)}\right]^2$$

$$\leq \; (a_1+a_2+a_3)[d(b_1,b_2)]^2$$
$$\Rightarrow [1-(a_1+a_2+a_3)][d(b_1,b_2)]^2 \leq 0,$$

Which is possible only if $[d(b_1,b_2)]=0 \Rightarrow b_1=b_2$ which gives us uniqueness of b.

Remark 15. If we take a_4=0, then we get Theorem (14) in [17]

Now we present the modified form of Theorem3 in terms of *b*-metric spaces.
Theorem 16.

Let (X, d) be a complete b-metric space with the coefficient $\delta \geq 1$ and suppose the self –maps T, g, h: $X \to X$ satisfy the condition

$$\begin{aligned}[d(Ta,gb)]^2 \;\leq\; & a_1[d(ha,Ta)d(hb,gb)]+ \\ & a_2[d(ha,gb)d(hb,gb)(1+d(Ta,hb))]+ \\ & a_3[d(ha,hb)d(hb,gb)(1+d(Ta,hb))]+a_4[d(ha,gb)d(hb,gb)] \text{-------(3.5)} \\ & a_5[\{d(hb,Ta)\}^2+\{d(hb,gb)\}^2]+a_6\left[\frac{d(hb,gb)+d(hb,ga)}{1+d(hb,Ta)d(ha,Ta)}\right]^2\end{aligned}$$

Where $a_1,a_2,a_3,a_4,a_5,a_6 \geq 0,$ are non negative real with

$$sa_1+(s^2+s)a_2+sa_3+(s^2+s)a_4+a_5+a_6<1$$

If T(X)$\cup g(X) \subseteq$ h(X) and h(x) is a complete subspace of X, then the maps T, g and h have a coincidence point u in X. Moreover, if {T, h} and{g, h} are weakly compatible pairs. Then T, g and h have a unique common fixed point in X.

Proof: Let $a_0 \in X$ *and defined the sequence* $\{b_n\}$ in X as follows,

$$b_{2n}=ha_{2n+1}=Ta_{2n}$$
$$b_{2n+1}=ha_{2n+2}=ga_{2n+1,} \;\; \forall n \geq 0$$

By using 3.5, we have

$$
\begin{aligned}
\left[d(b_{2n},b_{2n+1})\right]^2 &= \left[d(ha_{2n+1},ha_{2n+2})\right]^2 \\
&= \left[d(Ta_{2n},ga_{2n+1})\right]^2 \\
&\le a_1\left[d(ha_{2n},Ta_{2n})d(ha_{2n+1},ga_{2n+1})\right]+ \\
&\quad a_2\left[d(ha_{2n},ga_{2n+1})d(ha_{2n+1},ga_{2n+1})(1+d(Ta_{2n},ha_{2n+1}))\right]+ \\
&\quad a_3\left[d(ha_{2n},ha_{2n+1})d(ha_{2n+1},ga_{2n+1})(1+d(Ta_{2n},ha_{2n+1}))\right]+ \\
&\quad a_4\left[d(ha_{2n},ga_{2n+1})d(ha_{2n+1},ga_{2n+1})\right]+ \\
&\quad a_5\left[\{d(ha_{2n+1},Ta_{2n})\}^2+\{d(ha_{2n+1},ga_{2n+1})\}^2\right]+ \\
&\quad a_6\left[\frac{d(ha_{2n+1},ga_{2n+1})+d(ha_{2n+1},ga_{2n+1})}{1+d(ha_{2n+1},Ta_{2n})d(ha_{2n},Ta_{2n})}\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&\le a_1\left[d(ha_{2n},ha_{2n+1})d(ha_{2n+1},ha_{2n+2})\right]+ \\
&\quad a_2\left[d(ha_{2n},ha_{2n+2})d(ha_{2n+1},ha_{2n+2})(1+d(ha_{2n+1},ha_{2n+1}))\right]+ \\
&\quad a_3\left[d(ha_{2n},ha_{2n+1})d(ha_{2n+1},ha_{2n+2})(1+d(ha_{2n+1},ha_{2n+1}))\right]+ \\
&\quad a_4\left[d(ha_{2n},ha_{2n+1})d(ha_{2n+1},ha_{2n+2})\right]+ \\
&\quad a_5\left[\{d(ha_{2n+1},ha_{2n+1})\}^2+\{d(ha_{2n+1},ha_{2n+2})\}^2\right]+ \\
&\quad a_6\left[\frac{d(ha_{2n+1},ha_{2n+2})+d(ha_{2n+1},ha_{2n+1})}{1+d(ha_{2n+1},ha_{2n+1})d(ha_{2n},ha_{2n+1})}\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&\le a_1\left[d(b_{2n-1},b_{2n})d(b_{2n},b_{2n+1})\right]+ \\
&\quad a_2\left[d(b_{2n-1},b_{2n+1})d(b_{2n},b_{2n+1})(1+d(b_{2n},b_{2n}))\right]+ \\
&\quad a_3\left[d(b_{2n-1},b_{2n})d(b_{2n},b_{2n+1})(1+d(b_{2n},b_{2n}))\right]+ \\
&\quad a_4\left[d(b_{2n-1},b_{2n+1})d(b_{2n},b_{2n+1})\right]+ \\
&\quad a_5\left[\{d(b_{2n},b_{2n})\}^2+\{d(b_{2n},b_{2n+1})\}^2\right]+ \\
&\quad a_6\left[\frac{d(b_{2n},b_{2n+1})+d(b_{2n},b_{2n})}{1+d(b_{2n},b_{2n})d(b_{2n-1},b_{2n})}\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&\le a_1d(b_{2n-1},b_{2n})+sa_2\left[d(b_{2n-1},b_{2n})+d(b_{2n},b_{2n+1})\right]+ \\
&\quad a_3d(b_{2n-1},b_{2n})+sa_4\left[d(b_{2n-1},b_{2n})+d(b_{2n},b_{2n+1})\right]+a_5d(b_{2n},b_{2n+1})+ \\
&\quad a_6d(b_{2n},b_{2n+1})
\end{aligned}
$$

$$
\le (a_1+sa_2+a_3+sa_4)d(b_{2n-1},b_{2n})+(sa_2+sa_4+a_5+a_6)d(b_{2n},b_{2n+1})
$$

$$\left[1-(sa_2+sa_4+a_5+a_6)\right]d(b_{2n},b_{2n+1}) \le (a_1+sa_2+a_3+sa_4)d(b_{2n-1},b_{2n})$$

$$d(b_{2n},b_{2n+1}) \le \frac{a_1+sa_2+a_3+sa_4}{\left[1-(sa_2+sa_4+a_5+a_6)\right]}d(b_{2n-1},b_{2n})$$
$$d(b_{2n},b_{2n+1}) \le \lambda d(b_{2n=1},b_{2n}),$$

Where $\lambda = \dfrac{a_1+sa_2+a_3+sa_4}{\left[1-(sa_2+sa_4+a_5+a_6)\right]} < \dfrac{1}{s} < 1,$

$$d(b_{2n},b_{2n+1}) \le \lambda d(b_{2n-1},b_{2n})$$

Similarly, it can be shown that, $d(b_{2n+1},b_{2n+2}) \le \lambda d(b_{2n},b_{2n+1}).$

Therefore, for all $n \in N$ we can get

Now, for any m>n, we have

$$\begin{aligned} d(b_n,b_m) &\le d(b_n,b_{n+1}) + d(b_{n+1},b_{n+2}) + \dots\dots\dots + d(b_{m-1},b_m) \\ &\le \left[\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \dots\dots\dots + \lambda^{m-1}\right]d(b_0,b_1) \\ &\le \frac{\lambda^n}{(1-\lambda)}d(b_0,b_1) \end{aligned}$$

Therefore, from Lemma 13, we have

$$d(b_n,b_m) \le \frac{s\lambda^n}{(1-s\lambda)}d(b_0,b_1) \to 0 \ \ as \ m,n \to \infty$$

Where $s\lambda < 1.$ It follows that the sequence $\{b_n\}$ is a Cauchy sequence and by the completeness of X, $\{b_n\}$ converges to some $b \in X$.

Therefore,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} Ta_{2n} = \lim_{n\to\infty} ga_{2n+1} = \lim_{n\to\infty} ha_{2n+1} = \lim_{n\to\infty} ha_{2n+2} = b$$

Since h(x) is a complete subspace of X, there exists u, b$\in X$ such that hu=b. If gu$\neq$ b, using 3.5 we get

$$\begin{aligned} \left[d(Ta_{2n},gu)\right]^2 \le a_1&\left[d(ha_{2n},Ta_{2n})d(hu,gu)\right] + \\ a_2&\left[d(ha_{2n},gu)d(hu,gu)(1+d(Ta_{2n},hu))\right] + \\ a_3&\left[d(ha_{2n},hu)d(hu,gu)(1+d(Ta_{2n},hu))\right] \end{aligned}$$

$$+a_4\left[d\left(ha_{2n},gu\right)d\left(hu,gu\right)\right]$$

$$+a_5\left[\left\{d\left(hu,Ta_{2n}\right)\right\}^2+\left\{d\left(hu,gu\right)\right\}^2\right]+a_6\left[\frac{d\left(hu,gu\right)+d\left(hu,g_{2n}\right)}{1+d\left(hu,Ta_{2n}\right)d\left(ha_{2n},Ta_{2n}\right)}\right]^2$$

Taking limit as $n\to\infty$

$$\left[d\left(b,gu\right)\right]^2\le a_1\left[d\left(b,b\right)d\left(b,gu\right)\right]+a_2\left[d\left(b,gu\right)d\left(b,gu\right)\left(1+d\left(b,b\right)\right)\right]$$

$$+a_3\left[d\left(b,b\right)d\left(b,gu\right)\left(1+d\left(b,b\right)\right)\right]+a_4\left[d\left(b,gu\right)\left(b,gu\right)\right]$$

$$\le\left(a_2+a_4+a_5+a_6\right)\left[d\left(b,gu\right)\right]^2$$

$$\Rightarrow\left[1-\left(a_2+a_4+a_5+a_6\right)\right]\left[d\left(b,gu\right)\right]^2\le 0,$$

Which is possible only if $\left[d\left(b,gu\right)^2\right]=0\Rightarrow b=gu.$ It follow that

hu = b =gu

since g and h are weakly compatible, we have hgu = ghu and so

g b = h b

If $b\ne gb$, then by 4.2.4 we the following.

$$\left[d\left(Ta_{2n,},gb\right)\right]^2\le a_1\left[d\left(ha_{2n},Ta_{2n}\right)d\left(hb,gb\right)\right]+a_2\left[d\left(ha_{2n},gb\right)d\left(hb,gb\right)\left(1+d\left(Ta_{2n,},hb\right)\right)\right]$$

$$+a_3\left[d\left(ha_{2n},hb\right)d\left(hb,gb\right)\left(1+d\left(Ta_{2n},hb\right)\right)\right]+a\left[d\left(ha_{2n},gb\right)d\left(hb,gb\right)\right]$$

$$+a_5\left[\left\{d\left(hb,Ta_{2n}\right)\right\}^2+\left\{d\left(hb,gb\right)\right\}^2\right]+a_6\left[\frac{d\left(hb,gb\right)+d\left(hb,ga_{2n}\right)}{1+d\left(hb,Ta_{2n}\right)d\left(ha_{2n,},Ta_{2n}\right)}\right]^2$$

As $n\to\infty,$ we have

$$\left[d\left(b,gb\right)\right]^2\le a_1\left[d\left(b,b\right)d\left(gb,gb\right)\right]+a_2\left[d\left(b,gb\right)d\left(gb,gb\right)\left(1+d\left(b,gb\right)\right)\right]$$

$$+a_3\left[d\left(b,gb\right)d\left(gb,gb\right)\left(1+d\left(b,gb\right)\right)\right]+a_4\left[d\left(b,gb\right)d\left(gb,gb\right)\right]$$

$$+a_5\left[\left\{d\left(gb,b\right)\right\}^2+\left\{d\left(gb,gb\right)\right\}^2\right]+a_6\left[\frac{d\left(gb,gb\right)+d\left(gb,b\right)}{1+d\left(gb,b\right)d\left(b,b\right)}\right]^2$$

$$\leq [(a_5 + a_6)][d(b, gb)]^2$$

$$\Rightarrow [1 - (a_5 + a_6)][d(b, gb)]^2 \leq 0,$$

and the above inequality is a possible only if $[d(b, gb)]^2 = 0$

$$\Rightarrow b = gb$$

By using 3.6,we have

hb =gb = b.

Again , if $b \neq T$ by3,.5 we have the following.

$$[d(Tu, ga_{2n+1})]^2 = [d(Tu, b)]^2$$

$$\leq a_1[d(hu, Tu)d(ha_{2n+1}, ga_{2n+1})]$$

$$+ a_2[d(hu, ga_{2n+1})d(ha_{2n+1}, ga_{2n+1})(1 + d(Tu, ha_{2n+1}))]$$

$$+ a_3[d(hu, ga_{2n+1})d(ha_{2n+1}, ga_{2n+1})(1 + d(Tu, ha_{2n+1}))]$$

$$+ a_4[d(hu, ga_{2n+1})d(ha_{2n+1}, ga_{2n+1})]$$

$$+ a_5\left[\{d(ha_{2n+1}, Tu)\}^2 + \{d(ha_{2n+1}, ga_{2n+1})\}^2\right]$$

$$+ a_6\left[\frac{d(ha_{2n+1}, ga_{2n+1}) + d(ha_{2n+1}, gu)}{1 + d(ha_{2n+1}, Tu)d(hu, Tu)}\right]^2$$

Taking limit as $n \to \infty,$ we have

$$[d(Tu, b)]^2 \leq a_1[d(b, Tu)\, d(b, b)] + a_2[d(b, b)d(b, b)(1 + d(Tu, b))]$$

$$+ a_3[d(b, b)d(b, b)(1 + d(Tu, b))] + a_4[d(b, b)d(b, b)]$$

$$\leq a_5\left[d(b, Tu)^2\right]$$

$$\Rightarrow [1 - a_5]\left[d(b, Tu)^2\right] \leq 0,$$

And this is possible only if $[d(Tu, b)]^2 = 0 \Rightarrow b = Tu.$ since T and h are weakly compatible, Thu =hTu:

$$\Rightarrow \text{ Tb =gb}$$

By 3.7 and 3.8 we have

$$\Rightarrow Tb = gb = hb = b$$

Thus, b is the unique common fixed point of T, g and h.

Uniqueness. In order to see the uniqueness of the common fixed point, let b_1 and b_2 be two distinct common fixed points of the self –map T, g and h such that $b_1 \neq b_2$.

Then by using 3.5 we get

$$[d(b_1, b_2)]^2 = [d(Tb_{1,}, gb)]^2$$

$$\leq a_1[d(hb_1, Tb_1)d(hb_2, gb_2)]$$

$$+ a_2[d(hb_{1,}, gb_2)d(hb_{2,}, gb_2)(1 + d(Tb_1, hb_2))]$$

$$+ a_3[d(hb_1, hb_2)d(hb_2, gb_2)(1 + d(Tb_1, hb_2))] + a_4[d(hb_1, gb_2)d(hb_2, gb_2)]$$

$$+ a_5[\{d(hb_2, Tb_1)\}^2 + \{d(hb_2, gb)\}^2] + a_6\left[\frac{d(hb_2, gb_2) + d(hb_2, gb)}{1 + d(hb_2, Tb_1)d(hb_1, Tb_1)}\right]^2$$

$$\leq a_1[d(b_1, b_1)d(b_2, b_2)] + a_2[d(b_1, b_2)d(b_{2,}, b_2)(1 + d(b_1, b_2))]$$

$$+ a_3[d(b_1, b_2)d(b_2, b_2)(1 + d(b_1, b_2))] + a_4[d(b_1, b_2)d(b_2, b_2)]$$

$$+ a_5[\{d(b_2, b_1)\} + \{d(b_2, b_2)\}^2] + a_6[\frac{d(b_2, b_2) + d(b_2, b_1)}{1 + d(b_2, b_1)d(b_1, b_1)}]^2$$

$$\Rightarrow [1 - (a_5 + a_6)][d(b_1, b_2)]^2 \leq 0,$$

and the inequality is possible only if $[d(b_1, b_2)]^2 = 0 \Rightarrow b_1 = b_2$ and common fixed point is unique.

The following corollaries are drived from the theorem 16

Corollary 17. let(X, d) be a complete b- metric space with the coefficient $s \geq 1$

and T, g, h be a self –mapping of X into itself satisfying

$$[d(Ta, gb)]^2 \leq a_1[d(ha, Ta)d(hb, gb) + d(hb, gb)d(hb, gb)(1 + d(Ta, hb))]$$

$$a_3[d(ha, hb)d(hb, gb)(1 + d(Ta, hb)) + d(ha, gb)d(hb, gb)]$$

For all a, b $\in X$ and a $a_1, a_3 \geq 0,$ such that $a_1 + (s^2 + 1)a_3 \prec 1$ then T, g, h have a

Common fixed point in X.

Proof : putting in theorem 4.2.4, we get the required result.

Corollary 18. let (X, d)be a complete b- metric space with the coefficient $s \geq 1$ *and* $T, g, h: X \to X$ be a self-mapping of X into itself satisfying the inequality

$$[d(Ta, gb)]^2 \leq a_1[d(ha, Ta)d(hb, gb)]$$

$$+ a_2[d(ha, gb)d(hb, gb)(1 + d(Ta, hb))]$$

For all a, b $\in X$ *and* $a_1, a_2 \geq 0,$ *such that* $a_1 + (s^2 + s)a_2 \prec 1$ then T, g, h have a common fixed point in X.

Proof: putting $a_3 = a_5 = a_6 = 0$ in theorem 4.2.4, we get the required result`

Remark 19. corollary 17, which gives the result of Pachpatte

Remark 20. corollary 18, which gives the result of Fisher

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