IDEALIZATION OF A DECOMPOSITION THEOREM

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ABSTRACT: In 1986, Tong [13] proved that a function f :(X, τ) $\rightarrow$ (Y, φ) is continuous if and only if it is α-continuous and A-continuous. We extend this decomposition of continuity in terms of ideals. First, we introduce the notions of regular-I-closed sets, A_I-sets and A_I-continuous functions in ideal topological spaces and investigate their properties. Then, we show that a function f :(X, τ , I) $\rightarrow$ (Y, φ) is continuous if and only if it is α-I-continuous and A_I-continuous.

Keywords: α-continuous and A-continuous, α-I-continuous and A_I-continuous

Contents

1. Introduction

In 1992, Jankovic and Hamlet [9] have introduced the notion of I-open sets in ideal topological spaces. Abd EI - Monsef et al. [1] further investigated I-open sets and I-continuous functions. In 1999, Dontchev [3] introduced the notion of pre -I - open sets which is weaker than that of I-open sets and by using this set, he provided a decomposition of I-continuity. Hatir and Noiri [5] introduced the notions of B_I -sets, C_I -sets, α -I-sets, semi-I-sets and β -I - open sets to obtain decompositions of continuity.

In this paper, first, we introduce the notions of regular-I-closed sets, A_I-sets and A_I-continuous functions in ideal topological spaces and investigate their properties. Then, we show that a function f :(X, τ , I)$\rightarrow$(Y,φ) is continuous if and only if it is α-I-continuous and A_I-continuous.

2.Preliminaries

Throughout the present paper, spaces always mean topological spaces on which no separation property is assumed unless explicitly stated. In a topological space (X, τ), the closure and the interior of any subset A of X will be denoted by Cl(A) and Int(A), respectively. A subset A is said to be regular closed if A = Cl(Int(A)). An ideal is defined as a nonempty collection I of subsets of X satisfying the following two conditions: (1) If A$\in$I and B$\subset$A, then B$\in$I; (2) If A$\in$I and B$\in$I, then $A \cup B \in I$. Let (X, τ) be a topological space and I an ideal of subsets of X. An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ , I). For a subset A$\subset$ X, $A^*(I)$) = {x$\in$ X |U$\cap A$ A$\notin$I for each neighborhood U of x} is called the local function of A with respect to I and τ [10]. X^* is often a proper subset of X. The hypothesis X = X^* [7] is equivalent to the hypothesis $\tau \cap$I = ϕ [12]. The ideal topological spaces which satisfy this hypothesis are called Hayashi-Samuels spaces. We simply write A^* instead of A^*(I) in case there is no chance for confusion. For every ideal topological space (X, τ , I), there exists a topology τ^*(I), finer than τ, generated by β(I, τ) = {U\I|U$\in\tau$ and I$\in$ I}, but in general β(I, τ) is not always a topology [8]. Additionally, Cl*(A)=A$\cup$A* defines a Kuratowski closure operator for τ*(I).

The following lemma is useful in the sequel:

Lemma 1 [8]. Let (X, τ , I) be an ideal topological space and A, B subsets of X. Then the following properties hold:

a) If A$\subset$B, then A^* $\subset$ B*,
b) A* = Cl (A*)$\subset$Cl (A),
c) (A*)*$\subset$A*,
d) (A$\cup$ B)* =A*$\cup$ B*.

We recall some definitions used in the sequel.

DEFINTION 1. A subset A of a topological space (X, τ) is said to be

a) α-open [11] if A$\subset$Int (cl (Int (A))),
b) A-set [13] if A = U$\cap$V, Where U is open and V is regular closed,
c) Locally-closed [2] if A = U$\cap$V, Where U is open and V is closed,
d) α*- set [6] if Int (A) = Int (Cl (Int (A))),
e) C-set [6] if A = U$\cap$V, Where U is open and is an α*-set.

DEFINITION 2. A subset A of an ideal topological space (X, τ, I) is said to be

a) $*$ - dense - in - itself [7] if A$\subset$A*,
b) τ*-closed [8] if A*$\subset$A,
c) $*$-perfect [7] if A=A*,
d) Semi-I-open [5] if A$\subset$Cl*(Int(A)),
e) α-I-open [5] if A$\subset$Int (Cl* (Int(A)),
f) α*-I-open [5] if Int(A)=Int (Cl* (Int(A))),
g) C$_I$-set [5] if A = $U \cap V$, Where U is open and is α*-I-open,
h) Pre-I-open [3] if A$\subset$Int (cl*(A)),
i) I-open [9] if A$\subset$ Int (A*),
j) I-locally-closed [3] if A = U$\cap$V, Where U is open and V is $*$-perfect.

3. Regular-I-closed sets

DEFINITION 3. A subset A of an ideal topological space (X, τ, I) is said to be regular-I-closed if A = (Int (A))*.

We denote by R$_I$C(X,τ) the family of all regular-I-closed subsets of (X, τ, I), when there is no chance for confusion with the ideal.

PROPOSITION 1. For a subset A of an ideal topological space (X, τ, I), the following properties hold:

a) Every regular-I-closed set is α*-I-open and semi-I-open,
b) Every regular-I-closed set is *-perfect.

PROOF. a) Let A be a regular-I-closed set. Then, we have $cl^*(Int(A)) = Int(A) \cup (Int(A))^* = Int(A) \cup A = A$. Thus, $Int(Cl^*(Int(A))) = Int(A)$ and $A \subset Cl^*(Int(A))$. Therefore, A is α *-I-open and semi-I-open.

b) Let A be a regular-I-closed set. Then, we have $A=(Int(A))^*$. Since $Int(A) \subset A$, $(Int(A))^* \subset A^*$ by lemma 1. Then, we have $A=(Int(A))^* \subset A^*$. On the other hand, by lemma 1 it follows from $A=(Int(A))^*$ that $A^*=((Int(A))^*)^* \subset (Int(A))^*=A$. Therefore, we obtain $A=A^*$. This show that A is $*$ -perfect.

REMARK 1. The converses of proposition 1 need not be true as the following examples show.

EXAMPLE 1. Let X=a, b, c, d}, $\tau =\{\phi$,X, {a, c},{d},{a, c, d}} and I={ ϕ ,{c}, {d},{c, d}}.

1) Set A={a, b}. Then, A is an α *-I-open set which is not regular-I-closed. For $A=\{a, b\} \subset X$, Since $Int(A)=\phi$, $(Int(A))^* =\phi$ and hence $Cl^*(Int(A)) = Int(A) \cup (Int(A))^*= \phi$. Thus, we have $Int(cl^*(Int(A)))=\phi= Int(A)$ and hence A is an α *-I-open set. On the other hand, since $(Int(A))^*=\phi \neq \{a, b\} = A$, A is not regular-I-closed.
2) Set A={a, c}. Then, A is a semi-I-open set which is not regular-I-closed. For $A=\{a,c\} \subset X$, Since, $Int(A)=\{a,c\}$,$(Int(A))^*=\{a,b,c\}$ and hence $Cl^*(Int(A))=Int(A) \cup (Int(A))^*=\{a, b, c\} \supset \{a, c\}=A$. This shows that A is a semi-I-open set. On the other hand, $(Int(A))^*=\{a, b, c\} \neq \{a, c\}=A$ and hence A is not regular-I-closed.
3) Let X={a, b, c}, $\tau =\{\phi$,X,{a},{a, b}} and I={ ϕ,{a}, {b}, {a, b}}. Set A={c}. Then A is -perfect but not regular-I-closed. For $A=\{c\} \subset X$, $A^*=\{c\}=A$ and hence A is $*$ -perfect. On the other hand, since Int (A)= ϕ and I we have $(Int(A))^*=(\phi)^*=\phi \neq \{c\}=A$. This shows that A is not regular-I-closed.

COROLLARY 1. Every regular-I-closed set is τ *-closed and *-dense-in-itself.

PROOF. The proof is obvious from proposition 1.

PROPOSITION 2. In an ideal topological space(X, τ ,I), every regular-I-closed set is regular closed.

PROOF. Let A be any regular-I-closed set. Then we have $(Int(A))^*=A$. thus, we obtain that $cl(A)=cl((Int(A))^*)=(Int(A))^*=A$ by lemma1.Additionally, by lemma 1, we have $(Int(A))^* \subset Cl(Int(A))$ and hence $A=(Int(A))^* \subset Cl(Int(A)) \subset Cl(A)=A$. Then we have A= Cl (Int(A)) and hence A is a regular closed set.

REMARK 2. The converse of proposition 2 need not be true as the following example shows.

EXAMPLE 2. Let X= {a, b, c, d}, τ ={ϕ,X,{a, c},{d},{a, c, d}} and I={ϕ,{c},{d},{c, d}}. Set A={b, d}. then A is a regular closed set which is not regular-I-closed. For A= {b, d}$\subset$X, Since Int(A)={d}, Cl (Int(A)) =Cl({d}) = {b, d} = A and A is a regular closed set. On the other hand, since Int(A)={d} and {d} $\in$ I, we have $(Int(A))^*=(\{d\})^*=\phi \neq \{b, d\}=A$ and hence A is not regular-I-closed.

PROPOSITION 3. Let (X, τ ,I) be an ideal topological space and I=ϕ or N, where N is the ideal of all nowhere dense sets.

Then a subset A of X is a regular-I -closed set if and only if A is regular closed.

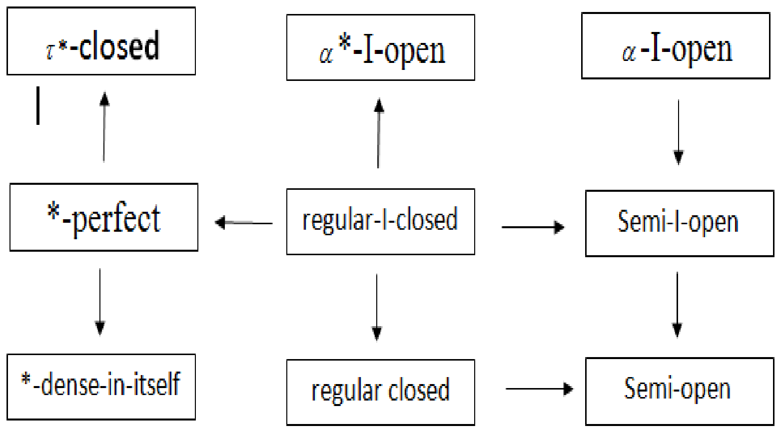
PROOF. By proposition 2, every regular-I-closed set is regular closed. If I= { ϕ }(resp. N), then it is well-known that $A^*=Cl(A)$ (resp. $A^*=Cl(Int(Cl(A)))$).Therefore , we obtain $(Int(A))^*=Cl(Int(A))$(resp.$(Int(A))^*=Cl(Int(Cl(Int(A))))=Cl(Int(A))$).Thus, regular-I-closed-ness and regular closedness are equivalent.

REMARK 3. Since every open set is α -I-open, regular-I-closedness and α -I-openness (and hence openness) are independent of each other as the following example shows.

EXAMPLE 3. In Example 1(2), A={a, c} is an open set but not a regular-I-closed. On the other hand, set A={a, b, c}. Then A is a regular-I-closed set which is not α -I-open. For A={a, b, c}$\subset$X, Int(A)={a, c} and $(Int(A))^*=\{a, b, c\}=A$. Hence A is a regular-I-closed set. On the other hand, since $(Int(A))^*=\{a, b, c\}$, we have $Cl^*(Int(A))=Int(A)\cup(Int(A))^*=\{a, b, c\}$ and $Int(Cl^*(Int(A)))= \{a, c\}$• {a, b, c}=A. Hence A is not α -I-open.

REMARK 4. For the relationship related to several sets defined above, we have the

Following diagram:



We can say that α *-I-openness and τ *-closedness are independent of each other. In Example 1(2), A= {a, c} is an α *-I-open set which is not τ *-closed. In Example 1, A= {b, d} is a τ *-closed set which is not α *-I-open. For, A*={b}$\subset${b, d}= A and A is τ *-closed. Moreover, Cl*(Int(A))=Cl*({d})= ϕ and hence Int(Cl*(Int(A)))= $\phi \neq${d}=Int(A). Therefore, A is not α *-I-open. Additionally, we can also say that regular closed and *-dense-in-itself are independent notions. In Example 1(2), A= {a, c} is a *-dense-in-itself set which is not regular closed. For, A*={a, b, c}$\supset${a, c}=A and Cl(Int(A))=Cl(A)={a, b, c}$\neq${a, c}=A. Moreover, A={b, d} is a regular closed set which is not *-dense-in-itself. We, recall that Hatir and Noiri [5] showed that α *-I-openness and semi-I-openness (resp. α -I-openness) are independent of each other.

4. A_I-sets

DEFINITION 4. A subset A of an ideal topological space (X, τ ,I) is called are A_I-set if A=U$\cap$V, where U$\in\tau$ and V$\in R_I C$(X, τ).

We denote by A_I(X, τ) the family of all A_I-sets of (X, τ ,I), when there is no chance for confusion with the ideal.

PROPOSITION 4:

Let (X,τ,I) be an ideal topological space and A a subset of X. Then the following properties hold.

a) If A is an open set and (X,τ,I) is a Hayashi-samuels space, then A

is an A_I-set,

b) If A is a regular-I-closed set, then A is an A_I-set.

PROOF:

Since $X \in \tau \cap R_IC(X,\tau)$,the proof is obvious.

REMARK 5:

The converses of proposition 4 need not be true as the following examples show.

EXAMPLE 4:

Let $X=\{a,b,c,d\}$, $\tau=\{X,\phi,\{a,c\},\{d\},\{a,c,d\}\}$ and $I=\{\phi,\{c\},\{d\},\{c,d\}\}$.

(1). Set $A=\{a,b,c\}$. Then A is an A_I-set but not open. For $A=\{a,b,c\} \subset X$, since $Int(A)=\{a,c\}$,$(Int(A))^*=\{a,b,c\}=A$ and hence A is a regular-I-closed set. Since $A=X \cap A$ and $X \in \tau$, A is an A_I-set. On the other hand, $Int(A)=$

$\{a,c\} \neq \{a,b,c\}=A$ and hence A is not open.

(2) . Set $A=\{a,c\}$, then by Example 1(2) A is not regular-I-closed. Set $V=$

$\{a,b,c\}$.Then by Example3,V is regular-I-closed and A is open.Therefore,

$A=A \cap V$ is an A_I-set.

PROPOSITION 5:

Let (X,τ,I) be an ideal topological space and A a subset of X. Then the following properties hold`:

a) If A is an A_I-set, then A is a C_I-set and I-locally-closed,

b) If A is an A_I-set, then A is an A-set.

PROOF:

This is an immediate consequence of proposition 1 and 2.

REMARK 6:

The converses of proposition 5 need not be true as the following examples show.

EXAMPLE 5:

In Example 1(1),A={a,b} is a C_I-set but not an A_I-set. For A={a,b}$\subset$X, we have already shown that A is an α^*-I-open set in

Example 1(1). We obtain that A is a C_I-set by using [5,proposition 3.2.c].

Also, we have already shown that A is not a regular-I-cloesd set and X is the only open set which contains A. Hence A is not an A_I-set. Further-

more, since A^*={a,b,c}$\neq$A, A is not *-perfect and consequently A is not

I-locally-closed.

(2). Let A={c}. Then by Example 1(3) A is *-perfect and not regular-I-

Closed. Therefore, A is I-locally-closed and not an A_I-set. Furthermore,

We can say that A is α^*-I-open by using [5,propositions 3.1 and 3.2].

Consequently, A is a C_I-set.

(3) . Let A={b,d}. Then by Example 2, A is a regular closed set which is not regular -I-closed.Therefore A is an A-set which is not an A_I-set.

PROPOSITION 6:

For a subset A of a Hayashi-samuels space (X, τ ,I),the following properties are equivalent:

a) A is an open set,

b) A is an α -I-open set and an A_I-set,

c) A is an pre-I-open set and an A_I-set.

PROOF:

a) $\Rightarrow$b). Let A be an open set. Hence A is an α -I-open set by[5].

On the other hand ,A =A$\cap$X,where A$\in\tau$ and X is a regular-I-closed set.

Hence A is an A_I-set.

b) $\Rightarrow$c). This is obvious since every α -I-open set is pre-I-open.

c) $\Rightarrow$a). Let A be pre-I-open and an A_I-set. Then A=U$\cap$V,

Where U$\in\tau$ and V$\in R_IC(X,\tau)$.since A is pre-I-open, we have A=U$\cap$V

$\subset$ Int(Cl*(U $\cap$ V)) $\subset$ Int(Cl*(U) $\cap$ Cl*(V)). By corollary 1,V is τ *-closed and Cl*(V)=V.Therefore,we have Int(Cl*(U)$\cap$Cl*(V))=Int(Cl*(U)$\cap$V)=

Int(Cl*(U)$\cap$Int(V)) and U$\cap$V$\subset$U$\cap$Int (Cl*(U))$\cap$Int(V)=Int(U$\cap$Cl*(U)$\cap$

V)=Int(U$\cap$V).consequently,we obtain U$\cap$V$\subset$Int(U$\cap$V) and A=U$\cap$V is open.

5. Idealization of a decomposition theorem

DEFINITION 5:

A function f:(X,τ ,I)$\rightarrow$(Y,φ) is said to be A_I-continuous(resp.

C_I-continuous[5], I-LC-continuous[3], A-continuous[13]) if for every

V$\in\varphi$,f^{-1}(V) is an A_I-set (resp.C_I-set,I-locally-closed set,A-set).

PROPOSITION 7:

For a function f:(X,τ ,I)$\rightarrow$(Y,φ), the following properties hold:

a) If f is A_I-continuous,then f is C_I-continuous.

b) If f is A_I-continuous,then f is I-LC-continuous.

c) If f is A_I-continuous, then f is A-continuous.

PROOF:

The proof is obvious from proposition 5.

REMARK 7:

The converses of proposition 7 need not be true as the following example shows.

EXAMPLE 6:

Let (X, τ ,I) be the same ideal topological spaces as in Example 1(1) and Example 2 for (1),(2) and (3),respectively. Let Y={a,b} and φ ={Y, ϕ ,{a}}.

(1) Let f:(X, τ ,I) $\rightarrow$ (Y, φ) be a function defined as follows: f(a)=f(b) =a And f(c)=f(d)=b. Then f is C_I-continuous but not A_I-continuous by Example 1(1).

(2) Let f:(X, τ ,I) $\rightarrow$ (Y, φ) be a function defined as follows: f(a)=f(b) =f(d)=b and f(c)=a. Then f is I-LC-continuous but not A_I-continuous by Example 5(2).

(3) f:(X, τ ,I) $\rightarrow$ (Y, φ) be a function defined by f(b)=f(d)=a and f(a) =f(c)=b.Then f is A-continuous but not A_I-continuous by Example 2.

DEFINITION 6:

A function f:(X, τ ,I) $\rightarrow$ (Y, φ) is said to be z-I-continuous[5] (resp. Pre-I-continuous [3]) if f^{-1}(V) is α -I-open (resp. Pre-I-open) in X For every open set V of (Y, φ).

THEOREM 1:

Let (X, τ ,I) be a Hayashi-samuels space. For a function f:(X, τ ,I) $\rightarrow$ (Y, φ), the following properties are equivalent:

a) f is continuous,

b) f is α -I-continuous and A_I-continuous.

c) f is pre-I-continuous and A_I-continuous.

PROOF:

This is an immediate consequence of proposition 6.

LEMMA 2 (Dontchev [3]):

Let (X, τ ,I) be an ideal topological space and I={ ϕ } or N. Then a subset A of X is pre-I-open if and only if A is preopen.

The following results are shown by Tong [13] and Ganster and Reilly [4] for the usual topological space.

COROLLARY 2:

Let (X, τ ,I) be an ideal topological space and I={ ϕ } or N. For a function f:(X, τ ,I) $\rightarrow$ (Y, φ), the following properties are equivalent:

a) f is continuous,

b) f is α -continuous and A-continuous (Tong [13]),

c) f is percontinuous and A-continuous (Ganster and Reilly[4]).

PROOF:

(1) Let I={ ϕ },we have A^*=Cl(A) and $Cl^*(A)=A\cup A^*$=Cl(A) for any Subset A of X. Therefore, we obtain (a) A is α -I-open if and only if it is α -open and (b) A is an A_I-set if and only if it is an A-set. The proof follows from Lemma 2 and Theorem 1 immediately.

(2) Let I=N, then we have A^*=Cl(Int(Cl(A))) and $Cl^*(A)=A\cup A^*=$ $A\cup$ Cl(Int(Cl(A))) for any subset A of X. Therefore,

$$\mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A)))=\mathrm{Int}[\mathrm{Int}(A)\cup \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))))]$$

$$=\mathrm{Int}[\mathrm{Int}(A)\cup \mathrm{Cl}(\mathrm{Int}(A))]$$

$$=\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))).$$

We obtain (a) A is α -I-open if and only if it is α -open and (b) A is an A_I-set if and only if it is an A-set. The proof follows from Lemma 2 and Theorem 1 imediately.

References

[1] M. E. Abd El-Monsef, E. F. Lashien and A. A. Nasef, on I-open sets and I-continuous functions, Kyungpook Math. J., **32** (1992), 21-30.

[2] N. Bourbaki, General topology, part I, Addison-wesley (Reading Mass., 1966).

[3] J.Dontchev, On pre-I-open sets and a decomposition of I-continuity, Banyan Math. J., **2**(1996).

[4] M. Ganster and I.Reilly, A decomposition of continuity, Acta Math . Hungar.,**56**(1990), 299-301.

[5] E. Hatir and T. Noiri, on decompositions of continuity via idealization, Acta Math. Hungar., **96**(2002), 341-349.

[6] E. Hatir, T. Noiri and S. Yuksel, A decomposition of continuity, Acta Math. Hungar., **70** (1996), 145-150.

[7] E Hayashi, Topologies defined by local properties, Math. Ann., **156** (1964), 205-215.

[8] D. Jankovic and T. R. Hamlett, New topologies from old via ideals, Amer. Math. Monthly, **97**(1990), 295-310.

[9] D. Jankovic and T. R. Hamlett, Compatible extensions of ideals, Boll. Un. Mat. Ital. **(7) 6-B** (1992), 453-465.

[10] K. Kuratowski, Topology, Vol. I, Academic Press (New York, 1966).

[11] O. Njastad, on some classes of nearly open sets, Pacific J. Math., **15** (1965), 961-970.

[12] P. Samuels, A topology formed from a given topological space, J. London Math. Soc. **(2), 10** (1975), 409-416.